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Hilbert C-Bimodules Given From Bundle Constructions*

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Abstract

This paper contains some remarks supplementary to the previous paper [KW] concerning Hilbert C*-bimodules given from bundle constructions and a example given by this construction concerning the product type action on C*-algebras.

KEYWORDS: Hilbert bimodule, bundle, tensor product.

1. Introduction

This paper contains a exposition of bundle construction of Hilbert C*-bimodules and some supplementary remarks about the previous paper[KW].

Kosaki-Yamagai[KoY] makes a construction of W*-bimodules associated with finite dimensional Hilbert bundles over discrete groups using finite groups, and Kajiwara-Yamagami[KaY] extends this construction to compact group case in the same W*-situation. This construction is useful for the presentation of new graphs about W*-index theory.

In [KW], we define the Hilbert C*-bimodules of finite type and define the bundle construction of Hilbert C*-bimodules using finite groups. In this paper, we state some results supplementary to [KW] and give an interesting examples concerning the product type action of finite group on a UHF algebra. This is considered in [KaY] in W*-situation.

In the last part, we state the remarks about Hilbert C*-bimodule theory and bundle constructions.

All C*-algebras appearing in this paper are assumed to be unital.

2. Hilbert C*-bimodules of finite type

We review the definitions and fundamental concepts of Hilbert C*-bimodules of finite type following [KW]. Let A and B be unital C*-algebras.

Definition [KW] *X is called a Hilbert A-B bimodule if the followings are satisfied.*

(1) X is a right pre-Hilbert B-module with $\langle x,y\rangle_B$,
(2) X is a left pre-Hilbert A-module with ${}_A\langle x,y\rangle$.
(3) Left A-action and right B-action commute each other.
(4) Two norms on X given by the two sided inner products are equivalent, and X is complete.
(5) The linear span of $\{{}_A\langle X,X\rangle\}$ is equal to A, and the linear span of $\{\langle X,X\rangle_B\}$ is equal to B.
*(6) The left representation of A and the right representation of B are bounded *-representations.*

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A finite subsets $\{u_i\}_{i=1,\dots,m}$ in X and $\{v_j\}_{j=1,\dots,n}$ in X are called a right B-basis for X and a left A-basis if for each $x \in X$ and

$$x = \sum_{i=1}^{m} u_i \langle u_i, x \rangle_B \qquad x = \sum_{i=1}^{n} {}_A\langle x, v_j \rangle v_j$$

Definition [KW] *Hilbert A-B bimodule X is called of finite type if X has a right B-basis and a left A-basis. We may define right and left index of X.*

Definition [KW] *Let X be a Hilbert A-B bimodule of finite type. For a right B-basis $\{u_i\}$ we put* $\mathrm{rind}[X] = \sum_i {}_A\langle u_i, u_i \rangle$ *and call this the right index of X. For a left A-basis $\{v_j\}$, we put* $\mathrm{lind}[X] = \sum_j \langle v_j, v_j \rangle_B$ *and call this the left index of X. When left or right index is a scalar, we put* $\mathrm{Ind}[X] = \mathrm{lind}[X] \cdot \mathrm{rind}[X]$ *and call this the index of X.*

If X and Y are Hilbert A-B bimodules of finite type. We put ${}_A \mathrm{Hom}_B(X, Y)$ be all linear maps from X to Y which commute with left-A and right B-actions. We put ${}_A \mathrm{End}_B(X) = {}_A \mathrm{Hom}_B(X, X)$. X is called irreducible if ${}_A \mathrm{End}_B(X)$ consists of only scalar operators.

3. Crossed products by Bundles

We review the definition of C*-crossed products by bundles following [KW]. Let G be a countable discrete group, H and K be finite subgroups of G. A finite dimensional Hilbert space V is called an H-K bundle over G if V is isomorphic to a direct sum $\oplus_{g \in G} V_g$ and there exist a unitary left H-action and a unitary right K-action on V such that $h \cdot V_g = V_{hg}$, $V_g \cdot k = V_{gk}$.

Let A be a unital C*-algebra and α be an action of G on A. Put $\hat{V} = A \otimes V$. Put $P = A \times_\alpha H$ and $Q = A \times_\alpha K$.

Then in [KW], $\hat{V}$ is made into a Hilbert P-Q bimodule, and the crossed product bimodule $\hat{V}$ is proved to be a Hilbert C*-bimodule of finite type.

The following is not stated in explicitly [KW] but very useful.

Proposition *H-K bundle V is itself a Hilbert $\mathrm{C}^*(H)$-$\mathrm{C}^*(H)$ bimodule of finite type. Moreover for any left $\mathrm{C}^*(H)$-basis $\{u_i\}$, the set $\{I \otimes u_i\}$ constitutes the left P basis of $\hat{V}$.*

Proof Action of group C*-algebras are the integrated form of groups actions. Two inner products are as follows.

$$\begin{aligned} {}_{\mathrm{C}^*(H)}\langle x, y \rangle(h) &= \sum_{g \in G} \langle x(g), h \cdot y(h^{-1}g) \rangle \\ \langle x, y \rangle_{\mathrm{C}^*(K)}(k) &= \sum_{g \in G} \langle x(g^{-1}), y(g^{-1}k) \cdot k^{-1} \rangle \end{aligned}$$

The formula for basis concerning the bundle V and the bimodule $\hat{V}$ is exactly same, and the second statement follows. □

By this, it is easily seen the fact that the indices of the bundle V and the bimodule $\hat{V}$ coincide, which is proved in [KW] directly.

Let $T \in {}_H \mathrm{End}_K({}_H V_K)$. We put $\hat{T}(a \otimes v) = a \otimes Tv$. Then $\hat{T}$ is in ${}_P \mathrm{End}_Q(\hat{V})$.

We assume that the action α of G on A is properly outer. We give the proof of the following proposition in [KW] whose proof are referred in [KW] to [KoY] where the proof of W*-case is presented.

Proposition *Let $\tilde{T}$ be in ${}_P \mathrm{End}_Q(\hat{V})$. Then there exists a unique $T \in {}_H \mathrm{End}_K({}_H V_K)$ such that $\tilde{T} = \hat{T}$.*

Proof. We take a CONS $\{v_i^g\}$ in V_g and fix them for each $g \in G$. We define $T_{i,j}^g \in \mathrm{End}(A)$ by

$$\tilde{T}(a \otimes v_i^g) = \sum_{g,j} T_{i,j}^g(a) \otimes v_j^g$$

We show that if $g \neq e$, $T^g_{i,j} = 0$ and if $g = e$, $T^e_{i,j}$'s are scalar operators. This finish the poof.

At first we show the following. Let $T \in \mathrm{End}(A)$. Then we show that for $g \neq e$, if (1) $T(aa') = aT(a')$ and (2) $T(a'a) = T(a')\alpha_g(a)$ for $a, a' \in A$ then T must be 0. By the first condition, there exists an $a'' \in A$ such that $T(a') = a'a''$. By the second condition, we have $aa'' = \alpha_g(a)a''$. Since α is assumed to be properly outer, $a'' = 0$.

By the condition $a\tilde{T}(a' \otimes v) = \tilde{T}(a(a' \otimes v))$ for $a, a' \in A$, we have the first condition for $T^g_{i,j}$'s for all $g \in G$, and by the condition $\tilde{T}(a' \otimes v)a' = \tilde{T}((a' \otimes v)a')$, we have the second condition for $T^g_{i,j}$'s for all $g \in G$. Then $T^g_{i,j}$'s are all 0 for $g \neq e$ and scalar operators for $g = e$. □

The categorical equivalence in [KW] is derived from this proposition easily.

We give an example concerning the inclusion of the fixed point algebras of the product type action of a finite group on UHF algebras. The same example in W*-situation is presented in [KaY]. We give the inner products of Hilbert C*-bimodules.

Example Let G be a finite group, W be a finite dimensional Hilbert space such that there exists a unitary representation π of G on it. We put $V = C(G) \otimes W$. We make this V into G-G bundle as follows.

$$g \cdot (w'_g \otimes v) = w_{g^{-1}g'} \otimes v \qquad (w_{g'} \otimes v) \cdot g = w_{g'g} \otimes \pi(g)v$$

Then the fusion rule of V is completely determined by the representation theory of G [KaY]. Let A be the UHF algebra which is given by the infinite tensor product of $\mathrm{End}(W)$ and α be the product type action of $\mathrm{ad}(\pi(g))$. Since the action α is outer, the fusion rule of $\hat{V}$ is also determined by the representation theory of G.

We put $Y = A \otimes W$. We make Y be into left A^α right $A \times_\alpha G$ bimodule. Left action is only multiplication on A. Right action is as follows.

$$(a \otimes w)a' = aa' \otimes w \qquad (a \otimes w)g = \alpha_{g^{-1}}(a) \otimes \pi(g^{-1})w$$

We define inner products as follows.

$$\begin{aligned} {}_{A^\alpha}\langle a_1 \otimes v_1, a_2 \otimes v_2\rangle &= \frac{1}{|G|}\sum_{g\in G} \alpha_g(a_1 a_2^*)\langle v_1, v_2\rangle \\ \langle a_1 \otimes v_1, a_2 \otimes v_2\rangle_{A\times_\alpha G} &= \frac{1}{|G|}\sum_{g\in G} a_1^*\alpha_{g^{-1}}(a_2)\langle \pi_{g^{-1}}(v_2), v_1\rangle\delta_g \end{aligned}$$

δ_g is the Dirac delta at g.

Then the left inclusion of Y is $A^\alpha \subset (A \otimes \mathrm{End}(W))^{\alpha\otimes\mathrm{ad}(\pi)}$. This is the C*-version of the inclusion given by Yin([Y]) for finite group G. On the other hand the right inclusion of Y is isomorphic to that of $\hat{V}$, and so the fusion rule of Yin's inclusion is completely determined by the representation theory of G.

4. Remarks

As for the example, in the contrast to the W*-case, when G is non finite compact group, the inclusion $A^\alpha \subset (A \otimes \mathrm{End}(W))^{\alpha\otimes\mathrm{ad}(\pi)}$ is not of finite type in Watatani's sense[W]. This fact is closely related to the infinite depth inclusions of factors and very mysterious, because the bundle $C(G) \otimes W$ is clearly has finite left basis and right basis.

In [KW], we treat only Hilbert C*-bimodules over unital C*-algebras. In non-unital case, it seems difficult to define the multiplier of bimodules where basis lies. It seems happen for some pathology concerning the crossed product construction of compact non-finite group actions.

In [KW] we treat the case all V_g's are Hilbert space, which are the special case of Hilbert C*-bimodules. It is also possible to treat the case V_g's are Hilbert C*-bimodules which have G-equivariant actions.

These matters will be studied in the forthcoming papers.

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